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On the Relation between White Shot Noise, Gaussian White Noise, and the Dichotomic Markov Process

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It is shown that the dichotomic Markov process converges to a white shot noise (interpreted according to the Stratonovich integration rule) in the joint limit in which the average duration of one of the states goes to zero and the value at this state goes to infinity. A further limit procedure allows us to obtain Gaussian white noise from white shot noise. These results are applied to the problem of noise-induced transitions. It is shown that white shot noise can give rise to transitions which do not occur for Gaussian white noise. The above results are finally generalized in introducing compound dichotomic Markov processes.

KEY WORDS: Dichotomic Markov process; white shot noise; noise-induced transitions.

1. INTRODUCTION

Gaussian white noise, shot noise and the dichotomic Markov process have all been widely used in the theory and applications of stochastic processes.(1–4)

Gaussian white noise (ξ_{GW}) is a purely random Gaussian process. Hence all its cumulants of order higher than 2 vanish identically(3) and the process is completely defined by

$$\langle \xi_{GW}(t) \rangle = 0 \tag{1}$$

$$\langle \xi_{GW}(t)\xi_{GW}(t') \rangle = 2D\delta(t - t') \tag{2}$$

We will refer to the parameter D as the strength of the noise.

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Shot noise (ξ_S, also called the Campbell's process) is defined as the sum

$$\xi_S(t) = \sum_i h(t - t_i) \tag{3}$$

where h is a given function and t_i are random time points. The probability to have such n time points in a time interval of duration t is thus given by

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \tag{4}$$

where λ stands for the given average density. In this paper, we will be concerned with white shot noise (ξ_{WS}), i.e., the case in which h is proportional to a Dirac δ function. White shot noise can thus be thought of as a sequence of δ peaks at random points in time, with a given average spacing. Possibly, the weight w of each pulse is an independent random variable defined by a probability density $\phi(w)$. In this case white shot noise is the time derivative of a compound Poisson process. Note that both white shot noise and Gaussian white noise are purely random processes but the former is non-Gaussian. White shot noise is thus characterized by the fact that its cumulants are (generally speaking) nonvanishing but δ correlated in time[5]:

$$\langle\langle \xi_{WS}(t_1) \cdots \xi_{WS}(t_n) \rangle\rangle \sim \delta(t - t_2) \cdots \delta(t_{n-1} - t_n) \tag{5}$$

In the case of the dichotomic Markov process, the random variable can only take two values Δ and Δ'. Each of these states has a given average duration τ_Δ and $\tau_{\Delta'}$, respectively, but the transition from one state to the other again occurs at random time points. The symmetric dichotomic Markov process, $\Delta = -\Delta'$ and $\tau_\Delta = \tau_{\Delta'}$, has the interesting property that it reduces to a Gaussian white noise in the limit $\Delta \to +\infty$ and $\tau_\Delta \to 0$ such that $(1/2)\Delta^2\tau_\Delta$ equals a constant D, which turns out to be the strength of the resulting Gaussian white noise.

In this paper, we will show that in another limit, the (asymmetric) dichotomic Markov process reduces to white shot noise. Indeed, it is expected intuitively that if the average duration of one state goes to zero, for instance, $\tau_{\Delta'} \to 0$, and the corresponding value Δ' goes to infinity such that the average weight $w_0 = \tau_{\Delta'} \times \Delta'$ remains constant, one recovers the picture of δ peaks at random time points, i.e., white shot noise. Furthermore, by letting the weights w_0 of the δ peaks go to zero, and simultaneously their density increase $\tau_\Delta \to 0$, such that w_0^2/τ_Δ remains constant, one recovers Gaussian white noise (see also Ref. 5).

The dichotomic Markov process (ξ_{DM}) is a simple example of a colored noise, its time autocorrelation function being an exponentially decreasing function with some finite correlation time τ_c:

$$\langle \xi_{\mathrm{DM}}(t)\xi_{\mathrm{DM}}(t')\rangle = \frac{D}{\tau_c}\exp\left(-\frac{|t-t'|}{\tau_c}\right) \tag{6}$$

In the limit $\tau_c \to 0$, one recovers the Gaussian white noise result (2).

The other simple example of a nonwhite (stationary) stochastic process with a time autocorrelation function of the type (6) is the Ornstein–Uhlenbeck process, which is the only stationary Markov stochastic process being moreover Gaussian.

Both of these processes have been used to investigate the effect of the coloration of the noise, in particular in the context of noise-induced transitions.[4,6–8] It was shown that colored noise can give rise to transitions which do not occur for Gaussian white noise. In this paper, we will also investigate the effect of the other basic type of white noise, namely, white shot noise, using the above-mentioned limit procedure.

It is known that in the case of multiplicative white noise, a stochastic equation does not have a meaning in itself. One should provide an integration rule. This is well known for the case of Gaussian white noise[3,9] but the same interpretation problem arises for white shot noise.[5] We will show that in the above-discussed limit, white shot noise has to be interpreted according to the Stratonovich integration rule.

In the above-mentioned limit, the dichotomic Markov process reduces to a white shot noise with exponentially distributed weights of the δ peaks. This is due to the fact that the duration of one state in a discrete Markov process is also exponentially distributed. In order to obtain a more general class of white shot noise processes, we introduce the compound dichotomic Markov process. This process differs from the usual dichotomic Markov process by the fact that the value Δ' assumed in one of the states is a random variable with probability density $\rho(\Delta')$. These compound processes allow us to generate white shot noise processes whose weight distribution $\phi(w)$ is essentially the convolution of a decaying exponential with ρ. However it turns out to be impossible to generate white shot noise with fixed weights of the δ peaks. This is due to the intrinsic stochasticity in the duration of one state of a discrete Markov process.

In Section 2, we prove the convergence of the dichotomic Markov process to white shot noise and of white shot noise to Gaussian white noise. These results are applied to the problem of noise-induced transitions in Section 3. In Section 4 it is proven that the Stratonovich integration rule applies. Particular examples are treated in Section 5. Finally, compound dichotomic Markov processes are introduced in Section 6 and their convergence to a large class of white shot noise processes is proven.

2. GAUSSIAN WHITE NOISE AND WHITE SHOT NOISE AS THE LIMITS OF THE DICHOTOMIC MARKOV PROCESS

We consider the dichotomic Markov process ξ_{DM} assuming the values Δ and Δ'. We denote by k_Δ and $k_{\Delta'}$ the transition probabilities per unit time between these two states, their average duration being $\tau_\Delta = 1/k_\Delta$ and $\tau_{\Delta'} = 1/k_{\Delta'}$, respectively. In the following we will always consider stochastic processes with vanishing average value. This implies for the dichotomic Markov process

$$\Delta\tau_\Delta + \Delta'\tau_{\Delta'} = 0 \tag{7}$$

ξ_{DM} is thus characterized by three independent parameters.

In order to investiate the convergence to a white shot noise process, we consider the time integral x of ξ_{DM}:

$$\partial_t x(t) = \xi_{DM}(t) \tag{8}$$

Since ξ_{DM} is not a white process, x will not be a Markov process.[10] However, the couple (x, ξ_{DM}) is a Markov process. The probabilities $P(x, \Delta, t)$ and $P(x, \Delta', t)$ to have x and $\xi_{DM} = \Delta$ or $\xi_{DM} = \Delta'$ at time t obey the following Master equations[4,6]:

$$\partial_t P(x, \Delta, t) = -\frac{\partial}{\partial x}\Delta P(x, \Delta, t) - k_\Delta P(x, \Delta, t) + k_{\Delta'} P(x, \Delta', t) \tag{9}$$

$$\partial_t P(x, \Delta', t) = -\frac{\partial}{\partial x}\Delta' P(x, \Delta', t) - k_{\Delta'} P(x, \Delta', t) + k_\Delta P(x, \Delta, t) \tag{10}$$

For the reduced probability $P(x, t) = P(x, \Delta, t) + P(x, \Delta', t)$, one obtains, taking as initial condition $P(x, \Delta', t = 0) = 0$ or $P(x, t = 0) = P(x, \Delta, t = 0)$,

$$\partial_t P(x, t) = -\frac{\partial}{\partial x}\Delta P(x, t) - \frac{\partial}{\partial x}(\Delta' - \Delta) \times \int_0^t \exp\left\{\left[-\Delta'\frac{\partial}{\partial x} - (k_\Delta + k_{\Delta'})\right](t - \tau)\right\} k_\Delta P(x, \tau)\, d\tau \tag{11}$$

This is a closed equation for the probability density $P(x, t)$, irrespective of the value of ξ_{DM}. Note that as a consequence of the elimination of the latter, Eq. (11) is no longer of a Markovian form.

Let us now consider the following limit:

$$\Delta' \to +\infty, \qquad k_{\Delta'} \to +\infty \tag{12}$$

with constant ratio $\Delta'/k_{\Delta'} = \Delta'\tau_{\Delta'} = w_0$.

In this limit, the dominant contribution to the integrand in (11) comes from times $\tau \simeq t$, hence (11) reduces to the following Markovian form:

$$\partial_t P(x, t) = -\frac{\partial}{\partial x}\Delta P(x, t) - \frac{\partial}{\partial x}\frac{w_0}{w_0(\partial/\partial x) + 1} k_\Delta P(x, t) \tag{13}$$

Let us now show that this limiting process x is equal to the integral y of a white shot noise ξ_{WS}:

$$\partial_t y(t) = \xi_{\mathrm{WS}}(t) \tag{14}$$

The stochastic equation (14) is equivalent with the following Master equation for the probability density $P(y,t)$ (see, for instance, Ref. 11):

$$\partial_t P(y,t) = -\frac{\partial}{\partial y}\Lambda P(y,t) + \lambda\left[\int_{-\infty}^{+\infty} P(y-w,t)\phi(w)\,dw - P(y,t)\right] \tag{15}$$

Here Λ stands for the constant value assumed by the white shot noise between the δ peaks, $\tau = 1/\lambda$ for the average time between two successive peaks, and $\phi(w)$ is the probability density that such a peak has a weight w (hence that the integral y changes abruptly by an amount equal to w). It is straightforward to see that Eqs. (13) and (15) are identical if we set

$$\Lambda = \Delta \tag{16}$$

$$\lambda = k_\Delta = \frac{1}{\tau_\Delta} \tag{17}$$

and

$$\phi(w) = \frac{\exp(-w/w_0)}{w_0}\theta(w) \qquad \text{if} \quad w_0 > 0 \tag{18a}$$

$$\phi(w) = -\frac{\exp(-w/w_0)}{w_0}\theta(-w) \qquad \text{if} \quad w_0 < 0 \tag{18b}$$

θ stands for the Heavyside function.

We have thus shown the convergence in law of x to the integral of white shot noise with an exponentially decaying distribution of the weights of the peaks. The exponential law is clearly a consequence from the fact that the duration of one state in a discrete Markovian process (in this case the state $\xi_{\mathrm{DM}} = \Delta'$) is also distributed exponentially.

It is interesting to recast the above limit in terms of other variables. We mentioned in the introduction that ξ_{DM} has a time autocorrelation function of the form (6). The correlation time τ_c is given by

$$\frac{1}{\tau_c} = \frac{1}{\tau_\Delta} + \frac{1}{\tau_{\Delta'}} \tag{19a}$$

and the noise strength reads

$$D = -\Delta\Delta'\tau_c \tag{19b}$$

Taking into account the condition (7) of vanishing average value, it remains to define a third independent parameter which we will call the "non-

Gaussianity" parameter γ:

$$\gamma = \frac{|\Delta - \Delta'|}{k_\Delta + k_{\Delta'}} = |\Delta - \Delta'|\tau_c \tag{19c}$$

The above-discussed limit is then equivalent to $\tau_c \to 0$ with fixed D and γ. This expresses that the limiting process, white shot noise, is a white process ($\tau_c = 0$) but it is still non-Gaussian ($\gamma \neq 0$). Note that the limit $\gamma \to 0$ can only be achieved in a nontrivial way by letting τ_c go to zero: the limit to a Gaussian process is only possible simultaneously with the limit to a white process. Hence it is impossible to obtain the Ornstein–Uhlenbeck process as a limit of the dichotomic Markov process.

Let us now prove that the limit to a Gaussian white noise can be taken also from a white shot noise by letting $\gamma \to 0$. White shot noise with exponentially distributed weights of the peaks in characterized by the parameters $\Delta = \Lambda$, which is the value assumed between two δ peaks, $\tau_\Delta = 1/\lambda = 1/k_\Delta$, which is the average time between two such peaks, and w_0, the average weight of each peak. The condition of vanishing average value implies

$$\langle \xi_{\mathrm{WS}} \rangle = \Delta\tau_\Delta + w_0 = 0 \tag{20}$$

Hence ξ_{WS} is defined by the following two independent parameters: a noise strength intensity D [compare with (19b)],

$$D = -\Delta w_0 = \frac{w_0^2}{\tau_\Delta} \tag{21a}$$

and a "non-Gaussianity" parameter [see also (19c)],

$$\gamma = w_0 \tag{21b}$$

The limit $\gamma \to 0$ with constant D is equivalent to letting the weights of the δ peaks go to zero, $w_0 \to 0$, and simultaneously increasing their density $\lambda = k_\Delta = 1/\tau_\Delta \to \infty$, such that $w_0^2\lambda = D$ remains constant. In this limit the Master equation (13) reduces to the following Fokker–Planck form:

$$\partial_t P(x,t) = D\frac{\partial^2}{\partial x^2} P(x,t) \tag{22}$$

Hence the integral of ξ_{WS} reduces to Brownian motion, which is the integral of Gaussian white noise.

The above results have been schematically represented in Table I. We have included in this table the Ornstein–Uhlenbeck process defined by the following stochastic differential equation:

$$\partial_t \xi_{\mathrm{OU}} = -\frac{\xi_{\mathrm{OU}}}{\tau_c} + \frac{\xi_{\mathrm{GW}}}{\sqrt{\tau_c}} \tag{23}$$

Table I. Gaussian White Noise, the Ornstein–Uhlenbeck Process, White Shot Noise, and the Dichotomic Markov Process. Classified by Following Important Properties and Related by Limit Theorems

	White process $\langle\xi(t)\xi(t')\rangle = 2D\delta(t-t')$		Colored process $\langle\xi(t)\xi(t')\rangle = \frac{D}{\tau_c}\exp[-\frac{\lvert t-t'\rvert}{\tau_c}]$
Gaussian process	Gaussian white noise ξ_{GW} parameters D	← $\tau_c \to 0$	Ornstein–Uhlenbeck process ξ_{OU} $\partial_t\xi_{OU} = -\frac{\xi_{OU}}{\tau_c} + \frac{\xi_{GW}}{\sqrt{\tau_c}}$ parameters D, τ_c
	↑ $\gamma \to 0$		
Non-Gaussian process	White shot noise ξ_{WS} $\langle\xi_{WS}\rangle = 0$ $\Delta\tau_\Delta + w_0 = 0$ parameters $\Lambda, \lambda, w_0, D, \gamma$	← $\tau_c \to 0$	Dichotomic Markov process ξ_{DM} $\langle\xi_{DM}\rangle = 0$ $\Delta\tau_\Delta + \Delta'\tau_{\Delta'} = 0$ parameters $\Delta, \Delta', \tau_\Delta, \tau_{\Delta'}, D, \tau_c, \gamma$

This process is determined by two independent parameters, the noise strength D and the correlation time τ_c. In the limit $\tau_c \to 0$, it reduces to Gaussian white noise.

3. NOISE-INDUCED TRANSITIONS

The above results can be of importance in the context of transitions under influence of external noise. Indeed, it is known that the macroscopic state of a system can undergo a qualitative change if one of the control parameters exhibits strong fluctuations.[4] It has been established that colored noise processes such as the dichotomic Markov process can give rise to transitions which disappear in the limit of Gaussian white noise. Let us now investigate the above-discussed limit in which the dichotomic Markov process reduced to the other fundamental white noise process, namely, white shot noise.

We consider the following external noise problem:

$$\partial_t x = f(x) + g(x)\xi_{DM} \tag{24}$$

Proceeding in the same way as in Ref. 6, one obtains the following stationary probability density $P_{st}(x)$ [taking into account (7)]:

$$P_{st}(x) = N \frac{g(x)}{[f(x) + \Delta g(x)][f(x) + \Delta' g(x)]} \times \exp\left\{ -\int_0^x \frac{(k_\Delta + k_{\Delta'}) f(x')}{[f(x') + \Delta g(x')][f(x') + \Delta' g(x')]} dx' \right\} \tag{25}$$

where N is a normalization constant. Note that the zeros of $f + \Delta g$ and $f + \Delta' g$ which correspond to the (supposedly unique) steady states of the evolution equation (24) for $\xi_{DM} = \Delta$ and $\xi_{DM} = \Delta'$, respectively, constitute the boundaries of the interval outside which $P_{st}(x) = 0$. The extrema of the probability density (25) are determined by the following equation:

$$f(x_M) + \frac{\Delta\Delta'}{k_\Delta + k_{\Delta'}} g(x_M) g'(x_M) + \frac{\Delta + \Delta'}{k_\Delta + k_{\Delta'}} f'(x_M) g(x_M) + \frac{1}{k_\Delta + k_{\Delta'}} \left[2f(x_M) f'(x_M) - \frac{f^2(x_M) g'(x_M)}{g(x_M)} \right] = 0 \tag{26}$$

The first term of (26) when set equal to zero is the equation for the deterministic steady state. In the limit of Gaussian white noise $\Delta = -\Delta'$, $k_\Delta = k_{\Delta'}$, and $\Delta \to +\infty$, $k_\Delta \to +\infty$ such that $\Delta^2/2k_\Delta = D$ the first two terms of (26) survive and one obtains the well-known result

$$f(x_M) - D g(x_M) g'(x_M) = 0 \tag{27}$$

In the limit of white shot noise $\Delta' \to +\infty$ and $k_{\Delta'} \to +\infty$ with constant ratio $\Delta'/k_{\Delta'} = w_0 = \gamma$, the first three terms of (26) equal to zero determine the extrema:

$$f(x_M) - D g(x_M) g'(x_M) + \gamma f'(x_M) g(x_M) = 0 \tag{28}$$

where we introduced the noise intensity D and "non-Gaussianity" parameter γ defined in (21). Of course, the result (28) reduces to (27) in the limit that $\gamma \to 0$. It is clear from a comparison of Eqs. (26) and (28) that transitions induced by white shot noise will also occur under the influence of (colored) dichotomic Markov noise, but the converse is not necessarily true. The term between brackets on the left-hand side of (26) is a correction term due to the existence of a finite correlation time [see formula (19)] in the dichotomic Markov process. In the same way, we conclude from a comparison of (27) and (28) that white shot noise can give rise to transitions which do not occur for Gaussian white noise. The last term on the left-hand side of (28) is a correction term due to the "non-Gaussianity" of white shot noise.

4. MULTIPLICATIVE WHITE NOISE IN STRATONOVICH INTERPRETATION

A stochastic differential equation with multiplicative white noise such as

$$\partial_t x = f(x) + g(x)\xi_{\mathrm{WS}} \tag{29}$$

is not defined in an unambiguous way. One has to provide an integration rule. The two most commonly used integration rules are the so-called Stratonovich and Ito integration rules.[9,12] In the previous section we considered the well-defined stochastic differential equation (24) with a multiplicative dichotomic Markov noise. The question arises according to which integration rule one has to interpret this equation if one takes the limit of ξ_{DM} going to ξ_{WS} as described in Section 2. According to Hänggi[5] (see also Ref. 14), Eq. (29) with ξ_{WS} a white shot noise with *fixed* weight w of the peaks is, in the Stratonovich interpretation, equivalent with the following Master equation:

$$\frac{\partial}{\partial t} P(x,t) = \left(-\frac{\partial}{\partial x}\left[f(x) - \lambda w g(x)\right] + \lambda\left\{\exp\left[-w\frac{\partial}{\partial x}g(x)\right] - 1\right\}\right)P(x,t) \tag{30}$$

For distributed weights, characterized by a probability density $\phi(w)$, this result is easily generalized. One obtains

$$\partial_t P(x,t) = \left(-\frac{\partial}{\partial x}\left[f(x) - \lambda\langle w\rangle g(x)\right] + \lambda\left\{\int \exp\left[-w\frac{\partial}{\partial x}g(x)\right]\phi(w)\,dw - 1\right\}\right)P(x,t) \tag{31}$$

In the case of weights distributed following the exponential law (18) the integral appearing on the right-hand side of (31) can be performed and the equation simplifies considerably:

$$\partial_t P(x,t) = \left\{-\frac{\partial}{\partial x}\left[f(x) - \lambda w_0 g(x)\right] - \lambda w_0 \frac{\partial}{\partial x} g(x)\frac{1}{w_0(\partial/\partial x)g(x) + 1}\right\}P(x,t) \tag{32}$$

For $f = 0$ and $g = 1$ one of course recovers Eq. (13). In fact the derivation done in Section 2 for this particular case can be repeated for general functions f and g. The Master equation for $P(x,t)$ obtained in the white shot noise limit turns out to be Eq. (32). We conclude that the multiplicative white shot noise obtained as the limit of the multiplicative dichotomic Markov process has to be interpreted in the Stratonovich sense. Note that in the Gaussian white noise limit $w_0 \to 0$, $\lambda \to \infty$ such that $w_0^2 \lambda = D$ remains constant, (32) reduces to

$$\partial_t P(x,t) = \left[-\frac{\partial}{\partial x} f(x) + D \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \right] P(x,t) \tag{33}$$

which is equivalent with the stochastic differential equation [ξ_{GW} being defined by (1) and (2)]:

$$\partial_t x = f(x) + g(x)\xi_{GW} \tag{34}$$

interpreted in the Stratonovich sense [see also Ref. 13]. Let us finally note that it seems impossible to obtain the stationary solution of (31) or even (30) in the general case. However, for exponentially distributed weights of the peaks, the stationary probability density is easily obtained from Eq. (32):

$$P_{st}(x) \sim \frac{1}{f(x) - \lambda w_0 g(x)} \exp\left\{ -\int^x \frac{f(x')}{[f(x') - w_0 g(x')] w_0 g(x')} dx' \right\} \tag{35}$$

It is easy to verify that (35) is the limiting form of the stationary probability density (25) for $\Delta' \to \infty$, $k_{\Delta'} \to \infty$ with fixed ratio $\Delta'/k_{\Delta'} = w_0$ [and $\lambda = k_{\Delta'}$, $\Delta = -\lambda w_0$, see (20)].

5. SOME EXAMPLES

In this section we discuss two examples of noise-induced transitions. Let us first consider a model introduced by Hongler[8]:

$$\partial_t x = -\tanh x + \frac{\xi}{\cosh x} \tag{36}$$

This model has the advantage that it can be linearized by a nonlinear transformation setting:

$$y = \sinh x \tag{37}$$

One obtains

$$\partial_t y = -y + \xi \tag{38}$$

The macroscopic equation

$$\partial_t \bar{x} = -\tanh \bar{x} \tag{39}$$

possesses the unique stable stationary solutions:

$$\bar{x}_M = 0 \qquad \text{(deterministic)} \tag{40}$$

For ξ equal to a Gaussian white noise, one obtains the extrema x_M [from Eq. (27)]:

$$\begin{aligned} x_M &= 0 \\ \cosh^2 x_M &= D \end{aligned} \qquad \text{(Gaussian white noise)} \tag{41}$$

where D is the strength of the Gaussian white noise. A bimodal probability density thus arises for $D > 1$. In the case of white shot noise defined by (18a), one obtains from (28)

$$\sinh^3 x_M + (1 - D)\sinh x_M + \gamma = 0 \qquad \text{(white shot noise)} \tag{42a}$$

This cubic equation possesses three real solutions for

$$D = \lambda w_0^2 \geqslant 1 + 3\left(\frac{\gamma}{2}\right)^{2/3} = 1 + 3\left(\frac{w_0}{2}\right)^{2/3} \tag{42b}$$

One can conclude by comparison with (41) that the deviation from a Gaussian law has the consequence that a transition will occur for higher values of the noise intensity.

The explicit expression for the stationary probability density can be obtained by performing the integration in (35). One obtains

$$P_{\text{st}}(x) \sim \frac{\cosh x}{(\sinh x + D/\gamma)^{1 - D/\gamma^2}} \exp\left(-\frac{1}{\gamma}\sinh x\right)\theta(x - x_B) \tag{43}$$

Note that the stationary probability density is identically zero for values smaller than the boundary value $x_B = \operatorname{arcsinh}(-D/\gamma)$, which is the value toward which x converges in between the δ peaks. This is a consequence of the asymmetry of ξ_{WS}, the pulses with positive weight w multiplied by $1/\cosh x$ being always positive (for $\gamma < 0$, i.e. for a distribution law (18b), x_B is an upper boundary for the stationary probability density). For values $D < \gamma^2$, the probability density diverges for $x \to x_B$, whereas it is zero at $x = x_B$ for $D > \gamma^2$. The various possible forms of the probability density have been represented in Fig. 1. Note that as $\gamma \to 0$, the boundary value x_B goes to $-\infty$ and one recovers the transition from a unimodal to a bimodal behavior at the value $D = 1$. For $\gamma \neq 0$, an important difference arises: at the crossing of the transition line (43), a second maximum of the probability density appears. In this sense, the "transition" induced by white shot noise is analogous to a first-order phase transition, whereas Gaussian white noise gives rise to a "second-order transition." Finally, in the case that ξ is the Ornstein–Uhlenbeck ξ_{OU} [see (23)] one knows that the process y, solution of the linear equation (38), is also a Gaussian process. It suffices to calculate its first and second moments in order to determine the latter completely. In particular, according to the transformation rule (37), one obtains the following stationary probability density $P_{st}(x)$ for the x variable:

$$P_{st}(x) = \frac{1}{2\pi[D/(1+\tau_c)]} \exp\left\{ -\frac{\sinh^2 x}{2\pi[D/(1+\tau_c)]} \right\} \cosh x \tag{44}$$

The extrema of this probability density are given by

$$\begin{aligned} x_M &= 0 \\ \cosh x_M &= \frac{D}{1+\tau_c} \end{aligned} \qquad \text{(Ornstein–Uhlenbeck process)} \tag{45}$$

This is completely analogous to the result (41) obtained for Gaussian white noise. In the example (36) the coloration of the Gaussian noise thus merely induces a shift in the transition point to a higher intensity $D = 1 + \tau_c$.

Let us now consider the following model of additive noise:

$$\partial_t y = -y + \xi \tag{46}$$

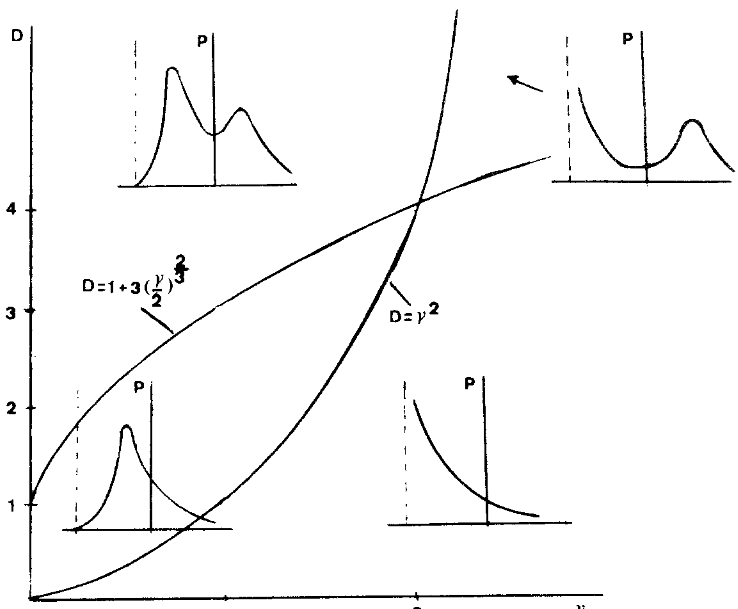

Fig. 1. Different forms of the stationary probability density for Hongler's model with multiplicative white shot noise.

The macroscopic equation has the unique stationary solution $\bar{y}_M = 0$. Adding a Gaussian white noise $\xi = \xi_{GW}$ does not alter this qualitative result: the stationary probability density is a Gaussian centered at $y_M = 0$. In the case of additive white shot noise $\xi = \xi_{WS}$ [with exponentially distributed weight (18a) and average waiting time $\tau = 1/\lambda$ between the peaks], the situation is more complex. The stationary probability density, obtained from (35), is the well known Γ-distribution

$$P_{st}(y) \sim (y - y_B)^{-1+(D/\gamma^2)} \exp\left(-\frac{y}{\gamma}\right) \theta(y - y_B) \tag{47}$$

with $y_B = -D/\gamma$. Note that the result (43) is of course recovered by applying the nonlinear transformation law (37). It is clear that the line $D = \gamma^2$ is a transition line between two different types of unimodal probability densities. For $D < \gamma^2$ P_{st} has a (normalizable) divergence at $y = y_B$, while for $D > \gamma^2$ $P_{st}(y_B) = 0$. According to (28), the probability density then has a maximum at $y_M = -\gamma > y_B$.

The above transition is a consequence of the asymmetry of the noise process. Nevertheless, a transition will also occur if the system is driven by a symmetric white shot noise with both positive and negative pulses for a sufficiently strong intensity D. This transition is toward a bihumped form of the probability density.

We can thus conclude that the "non-Gaussianity" of a white noise process can shift existing transitions and be responsible for new transitions.

6. THE COMPOUND DICHOTOMIC MARKOV PROCESS AND ITS WHITE SHOT NOISE LIMIT

The process which we will introduce now is still a (stationary) Markov process that can be in two states. The value assumed by the process in one state, which we will call the + state, is again equal to some constant Δ_+, but we assume that the value Δ_- assumed in the other state, the − state, is distributed according to a probability density $\rho(\Delta_-)$ (which is possibly a generalized function).[3] To make the distinction between the two states unequivocal, we also assume that ρ has no δ-Dirac contribution centered at the value Δ_+. We denote by $P_+(t)$ and $P_-(\Delta_-, t)$ the probability densities to be in the + state $\xi(t) = \Delta_+$ or in the − state with $\xi(t) = \Delta_-$. Let k_+ and k_- be the transition probabilities per unit time between the two states. The

[3] Of course, one could make the further generalization that Δ_+ is also an independent random variable with given probability density ρ_+, but we will not need this generalization in the context of a white shot noise limit.

Master equation then takes the following form:

$$\begin{aligned} \partial_t P_+(t) &= k_- \int_{-\infty}^{+\infty} P_-(\Delta_-, t)\, d\Delta_- - k_+ P_+(t) \\ \partial_t P_-(\Delta_-, t) &= -k_- P_-(\Delta, t) + k_+ \rho_0(\Delta_-) P_+(t) \end{aligned} \tag{48}$$

Note that the above process reduces to the usual dichotomic Markov process if one considers the probabilities $P_+(t)$ and

$$P_-(t) = \int P_-(\Delta_-, t)\, d\Delta_-$$

or for the particular case that $\rho_-(\Delta_-) = \delta(\Delta_- - \Delta')$. Let us now investigate to what kind of white shot noise the above process converges in the limit that the $-$ state corresponds to δ pulses in time. We thus consider the limit $k_- \to +\infty$ such that the weights $w = \Delta_-/k_-$ are distributed following a given weight function, i.e., the following limit exists:

$$\rho(w) = \lim_{k_- \to +\infty} k_- \rho_-(wk_-) \tag{49}$$

Proceeding in the same way as in Section 2, one obtains for the integral x of the compound dichotomic Markov process the following non-Markovian equation [compare with Eq. (11)]

$$\begin{aligned} \partial_t P(x,t) = &-\frac{\partial}{\partial x} \Delta_+ P(x,t) - \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} d\Delta_- (\Delta_- - \Delta_+) \\ &\times \int_0^t d\tau \Bigg[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(t-\tau)^n}{n!} \int d\Delta_1 \cdots d\Delta_n \\ &\times K(\Delta_-, \Delta_1) \cdots K(\Delta_{n-1}, \Delta_n) \Bigg] k_+ \rho_-(\Delta_n) P(x,t) \end{aligned} \tag{50}$$

where we have assumed $P_-(x, \Delta_-, t=0) = 0$ and we have introduced the kernel $K(\Delta_1, \Delta_2)$:

$$K(\Delta_1, \Delta_2) = \delta(\Delta_1 - \Delta_2)\left[\Delta_1 \frac{\partial}{\partial x} + k\right] + k_+ \rho(\Delta_2) \tag{51}$$

Let us now consider the above-mentioned limit. In this limit, the kernel (51) reduces to the following local form $(w_1 = \Delta_1/k_-)$:

$$k(\Delta_1, \Delta_2) = \left[w_1 \frac{\partial}{\partial x} + 1\right] \delta(w_1 - w_2) \tag{52}$$

and one obtains from (50) the following Markovian equation [compare with

Eq. (13)]:

$$\partial_t P(x,t) = -\frac{\partial}{\partial x}\Delta_+ P(x,t) - \frac{\partial}{\partial x}\int_{-\infty}^{+\infty} dw \frac{w}{w(\partial/\partial x)+1}\rho(w)k_+ P(x,t) \tag{53}$$

In order to compare this result with the Master equation for the integral y of a white shot noise process with weight distribution $\phi(w)$, we rewrite Eq. (15) as follows:

$$\partial_t P(y,t) = -\frac{\partial}{\partial y}\Lambda P(y,t) + \lambda\left\{\int_{-\infty}^{+\infty} dw \exp\left(-w\frac{\partial}{\partial y}\right)\phi(w) - 1\right\}P(y,t) \tag{54}$$

The equations (53) and (54) coincide if $\Delta_+ = \Lambda$, $k_+ = \lambda$, and

$$-\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} dw \frac{w}{w(\partial/\partial x)+1}\rho(w) = \int_{-\infty}^{+\infty} dw \exp\left(-w\frac{\partial}{\partial x}\right)\phi(w) \tag{55}$$

or, identifying the coefficients of the partial derivatives $\partial^n/\partial x^n$

$$\int dw\, \rho(w) w^n = \int dw\, \phi(w)\frac{w^n}{n!} \tag{56}$$

In the particular case of the usual dichotomic Markov process, $\Delta_+ = \Delta$ and $\rho_-(\Delta_-) = \delta(\Delta_- - \Delta')$; hence $\rho(w) = \delta(w - w_0)$. One recovers as limit process a white shot noise with exponentially distributed weights [see Eq. (18)]. The general solution of (56) reads

$$\phi(w) = \int_{-\infty}^{+\infty} \frac{\exp(-w/w_0)}{w_0}\left[\theta(w)\theta(w_0) - \theta(-w)\theta(-w_0)\right]\rho(w_0)\,dw_0 \tag{57}$$

as can be easily verified explicitly.[4]

This relation gives us the weight distribution function $\phi(w)$ of the white shot noise process which is the limit of the compound dichotomic Markov process characterized by the function $\rho(w)$. It is important to note that (57) cannot be inverted: it is not always possible to obtain a white shot noise with an *a priori* given function ϕ as a limit of a compound dichotomic Markov process. In particular, it is not possible to obtain a white shot noise with fixed weight w_f of the peaks, i.e., $\phi(w) = \delta(w - w_f)$. This is due to the intrinsic stochastic element present in the sojourn time of one state in a discrete Markov process.

[4] The uniqueness of the solution (57) follows from the requirements of normalization of ϕ. Indeed any solution other than (57) can only differ by a term proportional to $\delta(w)$, hence it will no longer be normalized.

To prove the above statement, let us suppose that there exists a function $\rho(w)$ such that

$$\int_{-\infty}^{+\infty} dw\, \rho(w) w^n = \frac{w_f^n}{n!} \tag{58}$$

and

$$\int_{-\infty}^{+\infty} dw\, \rho(w) = 1, \qquad \rho(w) \geqslant 0 \tag{59}$$

Then, for every $M > 0$, one has

$$\lim_{n \to +\infty} \int_{-\infty}^{+\infty} (Mw)^{2n} \rho(w)\, dw = \lim_{n \to +\infty} \left(\frac{Mw_f}{(2n)!} \right)^{2n} = 0 \tag{60}$$

This implies that $\rho(w)$ (being positive everywhere) must be zero for values of w such that $(Mw)^2 > 1$ or $|w| > 1/M$. Since M is arbitrary, $\rho(w)$ can only take a nonzero value in $w = 0$, but $\delta(w)$ is not a solution of (58).

7. CONCLUSION

In this paper, we have considered four important examples of stationary Markov processes and we have shown how they are interrelated. We have illustrated these results on the problem of noise-induced transitions. Problems which have been solved with a dichotomic Markov noise can immediately be reduced to probems involving white shot noise. Unfortunately, not every kind of white shot noise process is covered by this limit procedure. In particular, white shot noise with fixed value of the weights of the δ peaks cannot be obtained.

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